

Density function associated with a nonlinear bifurcating map

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2006 J. Phys. A: Math. Gen. 39 6851

(<http://iopscience.iop.org/0305-4470/39/22/003>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.105

The article was downloaded on 03/06/2010 at 04:35

Please note that [terms and conditions apply](#).

Density function associated with a nonlinear bifurcating map

A D Alhaidari

Shura Council, Riyadh 11212, Saudi Arabia
and

Physics Department, King Fahd University of Petroleum & Minerals, Dhahran 31261,
Saudi Arabia

E-mail: haidari@mailaps.org

Received 11 December 2005, in final form 13 April 2006

Published 16 May 2006

Online at stacks.iop.org/JPhysA/39/6851

Abstract

In the class of nonlinear one-parameter real maps, we study those with bifurcation that exhibits a period doubling cascade. The fixed points of such maps form a finite discrete real set of dimension $2^n m$, where m is the (odd) number of ‘primary branches’ of the map in the non-chaotic region and n is a non-negative integer. A new special representation of these maps is constructed that should give more insight into the physical interpretation and enhance their applications in mathematical physics and nonlinear dynamics. We associate with the map a nonlinear dynamical system whose Hamiltonian matrix is real, tridiagonal and symmetric. The density of states of the system is calculated and shown to have a band structure. The number of density bands is equal to $2^{n-1} m$ unless $n = 0$ in which case the density has m bands. The location of the bands is independent of the initial state. It depends only on the map parameter and whether the ordering of the fixed points in the set is odd or even. Polynomials orthogonal with respect to this density (weight) function are constructed. The logistic map is taken as an illustrative example.

PACS numbers: 02.30.Oz, 05.45.Ac, 82.40.Bj

1. Introduction

For a given real parameter λ we consider the nonlinear one-dimensional map f_λ whose infinite action takes a segment of the real line, $x \in [x_-, x_+]$, into a finite-dimensional real discrete set. This is the set of stable fixed points of the map, which is parameterized by λ . The dimension of the set depends on the range of values to which λ belongs. However, for other values of λ , such maps exhibit chaotic behaviour where the limit of the map does not exist but is usually bounded. In the present work we study these maps in the non-chaotic region. Typically, this class of nonlinear maps bifurcate at a given set of values of the parameter λ , called the

bifurcation points (also known as branch points). We restrict our investigation to those maps with bifurcation diagrams (the set of fixed points of the map for all values of λ) that exhibit period doubling cascade. In other words, the bifurcation diagram consists of branches that bifurcate simultaneously (at the same value of λ) into two branches and for several values of λ .

This subclass of nonlinear one-dimensional real maps has been studied extensively in the mathematics and physics literature. For an introductory review, one may consult any one of the large numbers of books on the subject. The list shown in [1] is only an example of those that have relevance to our present work. Our contribution here is to give a new and alternative realization of such maps that should give more insight into the physical interpretation and enhance their applications in mathematical physics and nonlinear dynamics. With each map f_λ in this subclass and for a given initial point (state) x_0 , which belongs to its domain $[x_-, x_+]$, we associate a real symmetric matrix which is tridiagonal by construction and parameterized by λ and x_0 . Consequently, one could interpret such Hermitian matrix as the matrix representation (in some complete basis) of the Hamiltonian of a dynamical system that corresponds to the map. The density of states of the system could thus be calculated using any one of the techniques that were developed for tridiagonal Hamiltonian matrices [2]. We show that this density has a band structure with forbidden gaps. The number of density bands equals half of the dimension of the set of fixed points of the map. Whereas the locations of the bands are independent of x_0 ; they only depend on λ and the ordering (odd or even) of the set of fixed points. Additionally, we use the theory of orthogonal polynomials and their three-term recursion relations associated with tridiagonal matrices to find the set of polynomials orthogonal with respect to this density (weight) function.

In the following section, we start by presenting the mathematical notation necessary for the definition of objects needed to establish our findings. The tridiagonal symmetric matrix associated with f_λ and x_0 is constructed. The corresponding density function will be evaluated using the resolvent operator of the system. The ‘logistic map’ is taken as an illustrative example and the density of states with single and multiple bands will be given. Polynomials of the first and second kind that are orthogonal with respect to these density (weight) functions are constructed in section 3.

2. Tridiagonal Hamiltonian and the density of states

Let $x_0 \in [x_-, x_+]$ and define $x_{n+1} = f_\lambda(x_n)$ for $n = 0, 1, 2, \dots$. Therefore, the set of fixed points of the map in a non-chaotic region is given by the finite multi-valued limit,

$$\lim_{n \rightarrow \infty} x_n = \{c_k(\lambda)\}_{k=1}^N. \quad (1)$$

The dimension of this set, N , and the values of its elements are independent of the initial seed x_0 . They depend only on the real parameter λ [1]. Let m be the number of ‘primary branches’ in the bifurcation diagram for the given non-chaotic region (i.e., the branches at the minimum value of λ in that region). Therefore, if the bifurcation period doubling process in the region takes place at the bifurcation points $\lambda = \lambda_1, \lambda_2, \lambda_3, \dots$ etc, then the number of branches in the cascade will be $m, 2m, 4m, \dots$ etc. Thus, the dimension of the set of fixed points of the map in the given non-chaotic region is $N = 2^n m$ for all λ in the range $\lambda_n < \lambda < \lambda_{n+1}$. Schematically, this process could be written as

$$x_0 \xrightarrow{f_\lambda} x_1 \xrightarrow{f_\lambda} x_2 \xrightarrow{f_\lambda} \dots \xrightarrow{f_\lambda} \left\{ \begin{array}{c} c_1 \\ \downarrow \\ c_2 \\ \downarrow \\ \vdots \\ c_N \end{array} \right\} \quad (2)$$

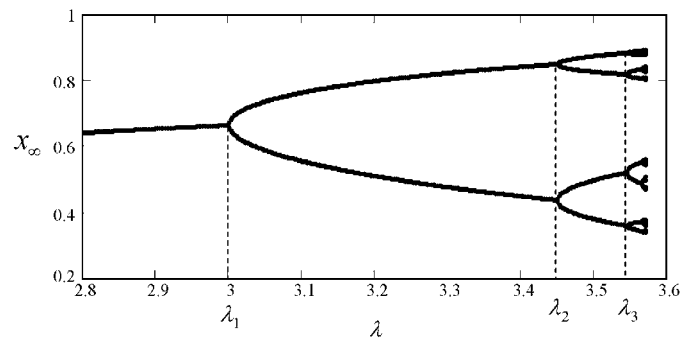


Figure 1. Bifurcation diagram of the logistic map in the non-chaotic region $2.8 < \lambda < 3.6$. The period doubling cascade is evident as well as the first few branch points at $\lambda = 3.00, 3.45, 3.54, 3.56$.

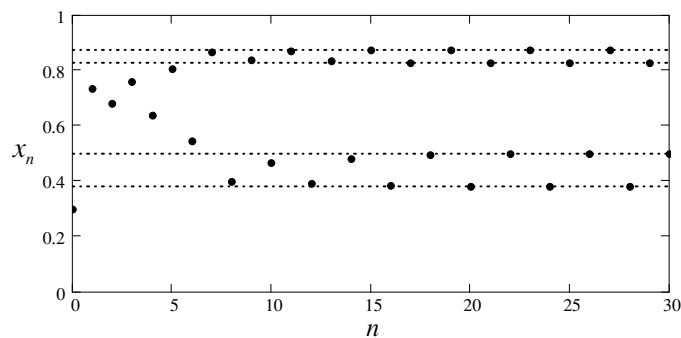


Figure 2. The action of the logistic map on the initial point $x_0 = 0.3$ in the region with period-4 orbits and where $\lambda = 3.5$. The four fixed points of the map are indicated by the horizontal dotted lines.

where the limiting action of the map cycles through the set of N fixed points $\{c_k\}_{k=1}^N$. As an illustration, we consider the well-known logistic map where $f_\lambda(x) = \lambda x(1-x)$, $x \in [0, 1]$ and $0 \leq \lambda \leq 4$. Figure 1 shows the bifurcation diagram of the map in the non-chaotic region $2.8 < \lambda < 3.6$ with one primary branch (i.e., $m = 1$). The period doubling cascade is evident as well as the first few bifurcation points, $\lambda_i = 3.00, 3.45, 3.54, 3.56$. The process, which is given schematically in (2), is shown graphically in figure 2 for the logistic map in the region with period-4 orbits and where $\lambda = 3.5$. It is also known that the logistic map has another non-chaotic region with period-3 ($m = 3$) doubling cascade in the neighbourhood of $\lambda \geq 3.828$.

Now if we define the set of real numbers $a_n = x_{2n}$ and $b_n = x_{2n+1}$ for $n = 0, 1, 2, \dots$ then we could construct the following real symmetric tridiagonal matrix:

$$H_{x_0}^\lambda = \begin{pmatrix} a_0 & b_0 & & & & & & & \\ b_0 & a_1 & b_1 & & & & & & \\ & b_1 & a_2 & b_2 & & & & & \\ & & b_2 & \times & \times & & & & \\ & & & \times & \times & \times & & & \\ & 0 & & & \times & \times & \times & & \\ & & & & & \times & \times & & \end{pmatrix}. \quad (3)$$

This construction could be displayed in a way that clearly shows the successive action of the map f_λ on x_0 as

$$H_{x_0}^\lambda = \begin{array}{c|c} \begin{array}{ccccccc} x_0 & \rightarrow & x_1 & & & & \\ & & \downarrow & & & & \\ x_2 & & & \rightarrow & x_3 & & 0 \\ & & & \downarrow & & & \\ & & x_4 & & & & \\ & & & \rightarrow & x_5 & & \\ & & & \downarrow & & & \\ & & x_6 & & & & \\ & & & \rightarrow & x_7 & & \\ & & & \downarrow & & & \\ & & & & x_8 & & \\ & & & & & & \\ & & & & & & \end{array} & \begin{array}{c} 0 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \\ \hline \begin{array}{c} 0 \\ \\ \\ \\ \\ \\ \\ \\ \end{array} & \end{array} \cdot \quad (3')$$

We refer to this construction as the ‘staircase’ construction¹. The infinitely far tail of this matrix will be made up of a repeated sequence (of length $N/2$) containing the set of fixed points of the map; that is,

$$\lim_{n \rightarrow \infty} a_n = \{c_{2i-1}(\lambda)\}_{i=1}^{N/2} \equiv \{\alpha_i\}_{i=1}^{N/2}, \quad \lim_{n \rightarrow \infty} b_n = \{c_{2i}(\lambda)\}_{i=1}^{N/2} \equiv \{\beta_i\}_{i=1}^{N/2}. \quad (4)$$

However, if the order of the fixed points in the set, $\{c_k\}_{k=1}^N$, is changed by a one-element shift to the right or left then α_i and β_i will be exchanged. We refer to the former ordering, which is given by (4), as even and to the latter as odd. The Hermitian matrix (3) is a faithful representation of the nonlinear map f_λ at x_0 since it contains all information about the action of the map at that point. On the other hand, $H_{x_0}^\lambda$ could also be interpreted as the matrix representation of a two-parameter Hamiltonian of the corresponding nonlinear system in the non-chaotic region. Changing x_0 will not affect the far tail of the tridiagonal matrix $H_{x_0}^\lambda$ except, possibly, for the ordering ambiguity. The resolvent operator (Green’s function) associated with $H_{x_0}^\lambda$ is formally written as $G_{x_0}^\lambda(z) = (H_{x_0}^\lambda - z)^{-1}$, which is an analytic matrix function in the complex z -plane except at the set of eigenvalues (discrete and continuous) of $H_{x_0}^\lambda$. One realization of the $(0, 0)$ component of this resolvent operator is given by the following infinite continued fraction [3, 4],

$$G_{x_0}^\lambda(z) = \frac{-1}{z - a_0 - \frac{b_0^2}{z - a_1 - \frac{b_1^2}{z - a_2 - \dots}}} \quad (5)$$

where, for simplicity of notation, we have omitted the $(0, 0)$ subscript on $G_{x_0}^\lambda$. This continued fraction could be approximated, for a given large enough integer M , as

$$G_{x_0}^\lambda(z) \cong \frac{-1}{z - a_0 - \frac{b_0^2}{z - a_1 - \frac{b_1^2}{z - a_2 - \dots - \frac{b_{M-2}^2}{z - a_{M-1} - T(z)}}}} \quad (6)$$

where the function $T(z)$, called the ‘terminator’, is written as

$$T(z) = \frac{\beta_1^2}{z - \alpha_1 - \frac{\beta_2^2}{z - \alpha_2 - \dots - \frac{\beta_{N/2}^2}{z - \alpha_{N/2} - T(z)}}}, \quad (7)$$

¹ Another unequivalent construction is the ‘zigzag’ construction where $a_n = x_{2n+1}$ and $b_n = x_{2n}$.

α_i and β_i are given by equation (4) in terms of the fixed points of the map at λ . A measure of the accuracy of the approximation (6) is given by how small are the deviations $|a_M| - |\alpha_1|$ and $|b_{M-1}| - |\beta_1|$. The integer M could, thus, be increased progressively to achieve the desired accuracy as well as calculation stability. The terminator $T(z)$ could easily be calculated using equation (7) once the set of coefficients α_i and β_i are determined from the fixed points. In fact, for small values of N , analytic expression for $T(z)$ could be obtained with little effort. As an example, for $m = 1$ and $N = 2$, we obtain

$$T(z) = \frac{1}{2}(z - \alpha_1) - \frac{1}{2}\sqrt{(z - \alpha_1 - 2\beta_1)(z - \alpha_1 + 2\beta_1)}. \quad (8)$$

The density of states $\rho_{x_0}^\lambda(y)$ of the dynamical system whose Hamiltonian is given by equation (3) could be obtained using Green's function $G_{x_0}^\lambda(z)$ as [2, 5, 6]

$$\rho_{x_0}^\lambda(y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} [G_{x_0}^\lambda(y + i\varepsilon) - G_{x_0}^\lambda(y - i\varepsilon)] = \frac{1}{\pi} \text{Im} [G_{x_0}^\lambda(y)], \quad (9)$$

where $y \in [y_-, y_+]$. Thus, the density is a measure of the discontinuity of the resolvent operator across the cut along the real line (named y because we have already used x to refer to the domain of the map) in the complex z -plane. This expression shows that the density vanishes wherever the value of the resolvent (on the real line) is real. Merging this fact with the approximation (6) we also conclude that the density vanishes wherever the value of the terminator $T(y)$ is real. Now, it is obvious that (7) gives a quadratic equation in $T(y)$. The solution of this equation for $T(y)$ results in a discriminant (the expression under the square root in the solution of the quadratic equation) which is a polynomial of degree N in y with real and λ -dependent coefficients. The reality and distribution of the fixed points $\{\alpha_i, \beta_i\}$ dictate that this polynomial has N real distinct roots, $y_1 < y_2 < \dots < y_N$ [3, 4, 7]. They are independent of x_0 and depend only on λ . Moreover, the value of this polynomial (the discriminant) between these roots alternate between positive and negative values. Thus, the value of $T(y)$ between these roots alternate between real and complex values. Consequently, the density structure will consist of $\frac{N}{2}$ non-zero bands interleaved with $\frac{N}{2} - 1$ forbidden gaps where the density vanishes. That is,

$$\rho_{x_0}^\lambda(y) = 0, \quad \text{for all } y \text{ in } y_{2i} < y < y_{2i+1}, \quad (10)$$

where $i = 1, 2, \dots, \frac{N}{2} - 1$, $y_- = y_1$ and $y_+ = y_N$. As a simple example, where $N = 2$, equation (8) gives the two roots $y_\pm = \alpha_1 \pm 2\beta_1$. The density of states has a single band with non-zero support between these two roots while vanishing outside. For $N = 4$, equation (7) results in four real roots as follows:

$$\frac{1}{2}(\alpha_1 + \alpha_2) + \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 \pm \beta_2)^2}, \quad (11a)$$

$$\frac{1}{2}(\alpha_1 + \alpha_2) - \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 \pm \beta_2)^2}. \quad (11b)$$

Thus, the density function is made up of two bands with a gap in between. The analytic expressions of the roots become much more complicated for higher values of N where one has to resort to numerical means to calculate them.

In the following graphical examples, we use the logistic map and work in the non-chaotic region where the bifurcation diagram (shown in figure 1) exhibits period doubling cascade. Additionally, the accuracy and stability of the density calculation are with respect to the choice $M = 100$ in equation (6). Figure 3(a) shows a single band density of the system before it gets into the cascade region and where $\lambda = 2.5$. On the other hand, figure 3(b) gives another single band density for a given initial state x_0 but the system is now in the period-2 bifurcation orbit where $\lambda_1 < \lambda = 3.2 < \lambda_2$. Figure 4(a) shows two double band densities for two different x_0 but with the same $\lambda = 3.5$. The system is in period-4 bifurcation orbit where $\lambda_2 < \lambda < \lambda_3$

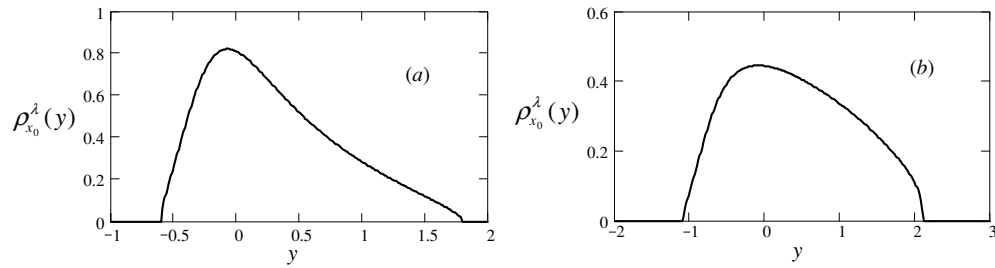


Figure 3. (a) Single band density associated with the map before bifurcation. We took $\lambda = 2.5$ and $x_0 = 0.3$. (b) Single band density associated with the map in the period-2 bifurcation orbit. We took $\lambda = 3.2$ and $x_0 = 0.4$.

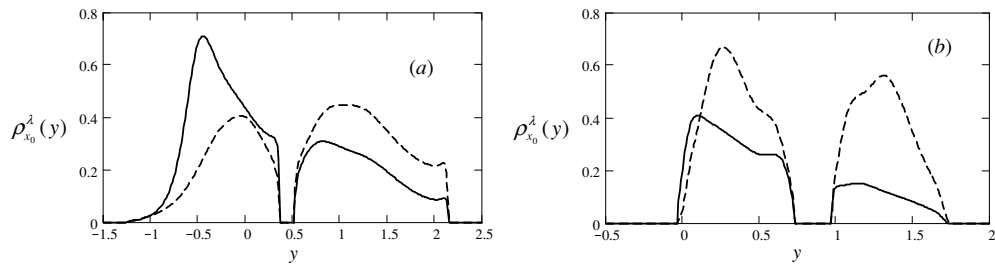


Figure 4. (a) Double band densities for two different x_0 but with the same $\lambda = 3.5$. The system is in period-4 bifurcation orbit and the ordering of the set of fixed points is even. The solid (dashed) curve corresponds to the initial point $x_0 = 0.30$ ($x_0 = 0.69$). (b) Same as (a) except for the ordering of the set of fixed points, which is odd. The solid (dashed) curve corresponds to the initial point $x_0 = 0.15$ ($x_0 = 0.73$).

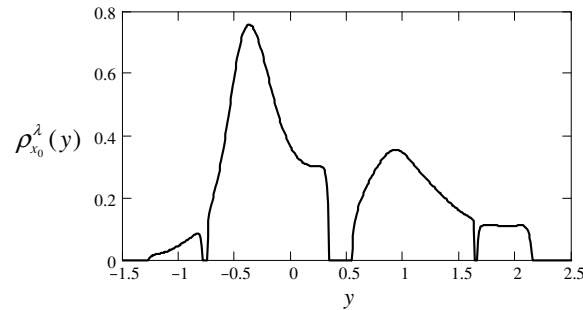


Figure 5. Four-band density associated with the logistic map in the period-8 bifurcation orbit. We took $\lambda = 3.55$ and $x_0 = 0.3$. The edges of the four bands are at $y = -1.267, -0.794, -0.743, 0.339, 0.546, 1.640, 1.665, 2.150$.

and the ordering of the set of fixed points is even. The same is repeated in figure 4(b) but for another two values of x_0 where the ordering of the same fixed points is odd. One should be able to observe the effect of the ordering on the location of the density bands due to the resulting parameter exchange $\alpha_i \leftrightarrow \beta_i$ as explained above. Figure 5 displays a four-band density associated with the system in the period-8 bifurcation orbit where $\lambda = 3.55$. For the sake of completeness, we plot in figure 6 the density of states of the system in the non-chaotic region with period-3 orbits ($m = 3$) for $\lambda = 3.835$ and $x_0 = 0.5$.

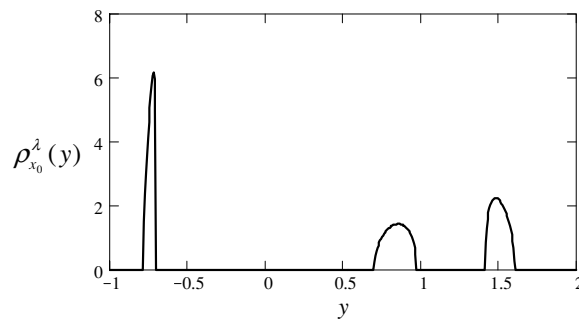


Figure 6. Density of states of the system in the non-chaotic region with period-3 orbit and for $\lambda = 3.835$, $x_0 = 0.5$. The edges of the three bands are located at $y = -0.783, -0.701, 0.701, 0.974, 1.414, 1.605$.

In the following section, we obtain the three-term recursion relation that defines the set of orthogonal polynomials associated with the nonlinear map f_λ at the given initial state x_0 . The weight function of these polynomials is the density of states found above.

3. Orthogonal polynomials associated with the map

The eigenvalue equation for the Hamiltonian $H_{x_0}^\lambda$ could be written as $H_{x_0}^\lambda |\psi\rangle = y |\psi\rangle$. We expand the eigenvector $|\psi\rangle$ in a complete orthogonal basis with the Fourier expansion coefficients $\{d_n(y)\}_{n=0}^\infty$. Assuming that the basis is chosen such that the matrix representation of the Hamiltonian $H_{x_0}^\lambda$ is tridiagonal and given by equation (3), then the eigenvalue equation results in the following three-term recursion relation for the expansion coefficients

$$y d_n(y) = a_n d_n(y) + b_{n-1} d_{n-1}(y) + b_n d_{n+1}(y), \quad n \geq 1 \tag{12}$$

with the initial relation, $n = 0$, $(a_0 - y)d_0(y) + b_0 d_1(y) = 0$. The solution of this recursion relation is written in terms of orthogonal polynomials [7, 8]. We refer to these polynomial solutions as $\{P_n^{(\lambda, x_0)}(y)\}_{n=0}^\infty$ and normalize them, as usual, by taking $P_0^{(\lambda, x_0)}(y) = 1$. Therefore, all of them become well defined just from knowledge of the recursion coefficients $\{a_n, b_n\}_{n=0}^\infty$, which are the matrix elements of the Hamiltonian. That is, we can write them as

$$\begin{aligned} P_0^{(\lambda, x_0)}(y) &= 1 \\ P_1^{(\lambda, x_0)}(y) &= \frac{y - a_0}{b_0} \\ P_2^{(\lambda, x_0)}(y) &= \frac{1}{b_1} [(y - a_1)P_1^{(\lambda, x_0)}(y) - b_0] \\ P_3^{(\lambda, x_0)}(y) &= \frac{1}{b_2} [(y - a_2)P_2^{(\lambda, x_0)}(y) - b_1 P_1^{(\lambda, x_0)}(y)] \\ P_4^{(\lambda, x_0)}(y) &= \frac{1}{b_3} [(y - a_3)P_3^{(\lambda, x_0)}(y) - b_2 P_2^{(\lambda, x_0)}(y)] \\ &\dots \end{aligned} \tag{13}$$

The theory of orthogonal polynomials associated with tridiagonal symmetric matrices shows that the weight function for these polynomials is the density function $\rho_{x_0}^\lambda(y)$ obtained above [3, 4]. That is, we can write the orthogonality relations as

$$\int_{y_-}^{y_+} \rho_{x_0}^\lambda(y) P_n^{(\lambda, x_0)}(y) P_m^{(\lambda, x_0)}(y) dy = \delta_{nm}. \tag{14}$$

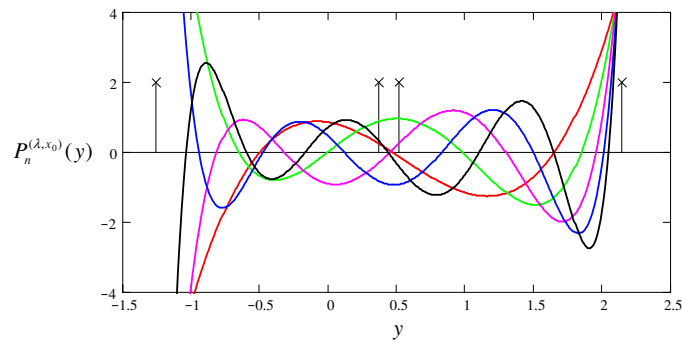


Figure 7. A plot of the orthogonal polynomials $\{P_n^{(\lambda, x_0)}(y)\}_{n=3}^7$ for $\lambda = 3.5$ and $x_0 = 0.5$ corresponding to the map in the period-4 bifurcation orbit. The stems with cross heads indicate the boundary of the non-zero support of the two-band weight (density) function.

(This figure is in colour only in the electronic version)

The integral breaks up into $N/2$ segments corresponding to the non-zero support (bands) of the density function. Figure 7 is a plot of the first few orthogonal polynomials $\{P_n^{(3.5, 0.3)}(y)\}_{n=3}^7$ associated with the nonlinear logistic map in the period-4 bifurcation orbit.

Polynomials of the second kind, which we refer to by $\{Q_n^{(\lambda, x_0)}(y)\}_{n=0}^\infty$, satisfy the same recursion relation (12). However, their corresponding initial relation ($n = 0$) is not the same. It is non-homogeneous and reads $(a_0 - y)Q_0(y) + b_0Q_1(y) = 1$. Moreover, $Q_0 = 0$. Thus, $Q_n^{(\lambda, x_0)}(y)$ is a polynomial with real coefficients and degree $n - 1$, whereas $P_n^{(\lambda, x_0)}(y)$ is of degree n . They satisfy a Wronskian-like relation (Liouville–Ostrogradskii formula) that reads

$$b_{n-1} [P_{n-1}(y)Q_n(y) - P_n(y)Q_{n-1}(y)] = 1, \quad n \geq 1. \quad (15)$$

Additionally, one can use these two kinds of polynomials to write an alternative representation for the $(0, 0)$ component of the resolvent operator as follows [3, 4]:

$$G_{x_0}^\lambda(z) = - \lim_{n \rightarrow \infty} [Q_n^{(\lambda, x_0)}(z) / P_n^{(\lambda, x_0)}(z)]. \quad (16)$$

In conclusion, we summarize our findings as follows. In the class of nonlinear one-parameter real maps, we considered those with bifurcation diagrams that consist of period doubling cascade. We associated with such a map a real tridiagonal symmetric matrix parameterized by the initial state x_0 and the map parameter λ . Interpreting this as the Hamiltonian matrix of the corresponding nonlinear dynamical system establishes a one-to-one correspondence between the nonlinear map and the density of states of the system. Our study showed that the density has a band structure. The number and location of these bands depended on the number and values of the fixed points of the map. They are also invariant with respect to variations in x_0 , except for a possible ambiguity in the ordering of the fixed points in the set. The logistic map was taken as an illustrative example where several density functions were given. Moreover, the orthogonal polynomials of the first and second kind associated with the nonlinear map were constructed. It is hoped that this study leads to further insight into the physical interpretation of such maps and enhances their applications in mathematical physics and nonlinear dynamics.

Acknowledgments

I am grateful to H A Yamani and M S Abdelmonem for motivating discussions and to H Bahlouli for the support in literature survey.

References

- [1] Banks J, Dragan V and Jones A 2003 *Chaos: A Mathematical Introduction* (Cambridge: Cambridge University Press)
- Seydel R 1994 *Practical Bifurcation and Stability Analysis: From Equilibrium to Chaos* 2nd edn (New York: Springer)
- Wiggins S 1990 *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (New York: Springer)
- Devaney R L 1989 *An Introduction to Chaotic Dynamical Systems* 2nd edn (Reading, MA: Perseus)
- [2] Yamani H A, Abdelmonem M S and Al-Haidari A D 2000 *Phys. Rev. A* **62** 052103
- [3] Akhiezer N I 1965 *The Classical Moment Problem* (Edinburgh: Oliver and Boyd)
- [4] Askey R and Ismail M 1984 *Recurrence Relations, Continued Fractions and Orthogonal Polynomials* (Providence, RI: American Mathematical Society)
- [5] Haymaker R W and Schlessinger L 1970 *The Padé Approximation in Theoretical Physics* ed G A Baker and J L Gammel (New York: Academic)
- [6] Pettifor D G and Weaire D L (ed) 1985 *The Recursion Method and its Applications* (Berlin: Springer)
- [7] Szegő G 1997 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)
- [8] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)